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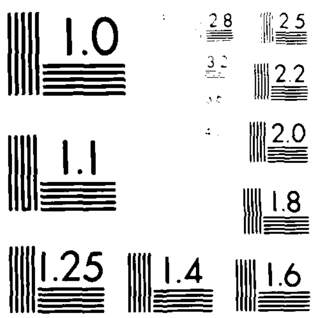

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IRREGULAR POINTS OF MODULATION

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IRREGULAR POINTS OF MODULATION

R. E. Meyer and J. F. Painter*

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ABSTRACT

The local theory of singular points is extended to a large class of linear, second-order, ordinary differential equations which can be physical Schroedinger equations or govern the modulation of real oscillators or waves. In addition to Langer's fractional turning points, such equations admit highly irregular points at which the coefficients of the differential equation can be almost arbitrarily multivalued. Genuine coalescence of singular points, however, is not considered. A local representation of solution structure is established which generalizes Frobenius' method of power series. Some results on solution symmetry have striking, global implications in the shortwave limit.

AMS (MOS) Subject Classifications: 34E20, 41A60, 30E15.

Key Words: Schroedinger equation, oscillator modulation, irregular singular point

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

This work concerns the modulation of waves or oscillating systems, which pervade all the science and engineering disciplines. Modulation occurs when waves travel through an inhomogeneous material in which the local propagation velocity differs from place to place, but the differences are small over a distance of only a wavelength -- a very common case in the sciences and engineering. The resulting change to the waves is mostly gradual, but occasionally drastic, as at a shadow-boundary, where oscillation turns into decay and quiescence over just a few wavelengths. When this phenomenon can be analyzed via an ordinary differential equation, such a boundary is called a transition point.

At first, only the simplest transition points representing the most typical shadow boundaries were studied. But then some phenomena, such as wave reflection and scattering cross-sections, came to be traced to hidden transition points that become visible only when real distance (or time) is embedded in its complex plane. When the material properties vary in a general manner, (which can often be observed only incompletely) the hidden transition points can have arbitrarily complex structure. The following work explores that structure in detail in order to contribute to the technical basis for a reform of the theory that will make it simpler and more and will furnish the tools for more efficient scattering calculations.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

IRREGULAR POINTS OF MODULATION

R. E. Meyer and J. F. Painter*

1. Introduction

The general linear second-order equation

$$\varepsilon^2 d^2w/dz^2 + q^2 w(z) = 0 \qquad (1)$$

with parameter ε and analytic coefficient function $q(z)$ is central to a vast class of problems in the sciences. In turn, it has long been appreciated that an understanding of the singular points and turning points of (1) is central to the treatment of those problems, regardless of whether those points occur at real z. Despite many studies, no complete theory of such points has been achieved, perhaps, because (1) encompasses too many disparate phenomena for a useful theory convering them all.

The present study focusses on those forms of (1) which can describe the physical modulation of waves or oscillators. To attempt only one step at a time, moreover, it excludes questions involving genuine coalescence of singular points and also singular points of (1) artificially introduced as representations of radiation conditions. This leaves a large class for study, none the less, because the coefficient functions $q(z)$ of (1) in the sciences must be defined, if not by speculation, then by measurement, in which case they can be known only imperfectly. The characterization of $q(z)$ cannot therefore be very specific and arbitrarily irregular points must be admitted on physical grounds, particularly when they do not occur at the real values

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of z of direct physical meaning. Certainly, multivalued functions $q(z)$ must be the norm, rather than the exception. All the same, modulation implies a certain structure for $q(z)$ (Section 2).

The main objective of the following is to extend to the general modulation equation the insight familiar from the theory of regular points that the local structure of the singularity can be expressed by a fundamental solution pair (w_m, w_s) of (1) associated with different 'exponents' or 'indices'. Thus, $w_m/w_s \to 0$ as the singular point is approached, so that $w_m(z)$ has a milder singularity than $w_s(z)$. The structure of irregular points can be described similarly by distinct singularities, one stronger (Section 4) and the other, milder (Section 3), which are associated with a fundamental solution pair constructed below.

In the general case of modulation, the coefficient function $q(z)$ is so poorly specified that the characterization of singularity structure cannot be very specific either. For more concrete results, the general class of such equations is therefore slightly restricted (Section 2) by the hypothesis that a limit of

$$q^{-2}(dq/dz) \int_0^z q(t)\,dt$$

be identifiable as the singular point $z = 0$ is approached. This still admits a much larger class than treated before [1] and makes possible a representation extending the familiar one for regular points in terms of power series commonly associated with the name of Frobenius [2]. The representation established below is in terms of quasi-power-series with coefficients that are themselves multivalued functions. The results reported concentrate on certain aspects of those functions and on solution symmetries which are important for the applications to scattering theory that motivate the present investigation.

A clearer indication of the structure thus revealed will be aided by a comment on the role of the parameter ε in (1), which is proportional to Planck's h in the quantum-mechnical interpretation of that equation. The results are local, in the first place, and as such, concern the "parameter-less case" in which ε is merely a fixed complex number (say, unity, without loss of generality). Since all the proofs are on that basis, ε could in fact be regarded as excess baggage. Quite the opposite is true, however, because $|\varepsilon|$ will reveal itself as a homotopy parameter linking irregular points to regular ones and thereby, as a signpost to a broad avenue of approach to a general theory of irregular points. For modulation equations, moreover, ε can also serve as an illuminating label for structural distinctions even in the parameter-less case. They reflect a distinction in the scientific meaning of the two terms of (1): the first -- or more precisely, $w^{-1}d^2w/d(z/\varepsilon)^2$ -- represents the oscillatory mechanism of Newton's law; by contrast, $q^2(z)$ represents the material potential. This is a little too naive, more rational variables x and εx must be introduced (Section 2) in the place of z/ε and z, respectively, but those do turn out to play fundamentally different roles of oscillation and modulation variables even in the parameter-less case. A third reason for treasuring ε is that the structural results have a striking significance for the shortwave case $|\varepsilon| << 1$.

The distinction between the modulation role of z and oscillation role of z/ε in (1) is an echo of the distinction between 'gravitational' and 'inertial' mass discussed by philosophers of physics in the late nineteenth century. It is therefore relevant to report that our comment on the benefits of ε is purely mathematical hindsight. The present investigation set out merely to establish semi-abstract existence constructions adequate to support a minimal connection theory [3] needed for scattering. That construction revealed a striking 'two-scale' structure in terms of functions of x and functions of εx. The functions of x are related to powers; those of εx, by contrast, have a structure peculiarly different from powers, they are multivalued 'mild' functions (Section 2) of a very general type including logarithms as familiar examples. A two-scale structure is thereby really implied only for the asymptotic representation (Section 5), which is remarkable in providing information global in x, even though only local in εx. Even in the parameter-less case, however, where our results are entirely local, the label ε helps to recognize the distinction between the role of the mild functions and the power functions in the solution structure.

For regular points, certain exponent differences lead to the "Frobenius exceptions" where the stronger solution contains logarithmic multiples of the milder one. Analogous features arise for irregular points (Section 4), where they are less pronounced because the structure is already pervaded by the 'mild', multivalued functions. However, they do spoil the symmetry of the stronger solution (Section 5) for a non-generic subset of equations.

One would naturally wish to extend to general irregular points Langer's [1] triumph of uniform approximation of solutions. It will indeed emerge strikingly below how general modulation equations are a direct extension of Bessel's equation. Beyond his class of fractional turning points, however, Bessel functions cannot serve as uniform approximands, because the branch points are more complicated, and it is not readily apparent yet how far the notion of uniform approximation could be extended to more general classes in a practically useful way. To some degree, at least, the 'local' representations established below do provide such an extension because they are global enough to answer asymptotic questions [3].

Good error bounds for the approximations proved below are also very desirable, but it appears premature to consider them in detail in this first foray. There are many indications that the formulation and results developed below are not definitive, but merely document an open road not noticed before. As it is explored further, better estimates will be found. Even the present ones, however, are based on the method of Volterra equations normally employed [4] to obtain error bounds, and to promote access to them, the symbol δ will be used to denote the elements from which such bounds wuld have to be assembled.

2. Modulation Equations

Equation (1) is only one of a family of normal forms of the general, linear, second-order differential equation and constructive statements are awkward and cumbersome in such an indefinite frame. By contrast, the Lionville-Green or WKB or Langer variable x such that

$$dx/dz = iq/\varepsilon$$

has long been recognized as the natural one for the description of waves or oscillators; it measures length or time in units of $(2\pi$ times) the local wavelength or period, and physical specifications, e.q., radiation conditions for scattering, relate directly to x or εx. Any simple and practical theory should emerge in terms of these variables, and continued reference to the independent variable z can only obscure matters and will be abandoned after this Section.

For $y(x) = w(z)$, (1) transforms into

$$y'' + 2fy' = y, \quad 2f(x) = -i\varepsilon q^{-2} dq/dz \quad , \tag{2}$$

which shows modulation to be controlled by the function $f(x)$, rather than by $q(z)$ directly. It explains also why roots of $q(z)$ ("turning points") which are ordinary points of (1) play the same role in the theory [4] as singular points of $q(z)$; all are singular points of the modulation function $f(x)$ and hence, are the singular points of (2) or of any other rational formulation of (1) as an equation describing modulation of waves.

Since genuine coalescence of singular points [5] is excluded from consideration here, each can be studied separately. This does not exclude functions $q(z;\varepsilon)$ in (1) with pairs or clusters of singular points approaching each other as $|\varepsilon|$ decreases, provided only that they do not do so too rapidly for a rescaling [6] to exist into a formulation (1) in which they remain bounded apart independently of ε. In order not to overload the presentation here, the rescaling [6] is presumed a priori, and the remaining dependence of q on ε (which is then immaterial for what follows) is ignored. Attention may then be directed to a single, singular point with an ε-independent neighborhood N free of others.

To define the subclass of equations (1) that can describe wave modulation, two requirements now suffice. First, the natural variable x must be defined, for otherwise, not even the concepts of wavelength or period could exist for (1). Secondly, if $q(z)$ be non-integrable at a singular point, then that point corresponds to no $\varepsilon x \in \mathbb{C}$ and hence represents not a genuine singularity of modulation, but a device for re-interpreting a radiation condition as a singular point in the z-plane. This will be excluded here to concentrate on the class of genuine modulation equations. For it, the singular point z must correspond to a definite point εx, and without loss of generality, both may be identified with the origin, so that

$$\varepsilon x = i \int_0^z q(t)\,dt \tag{3}$$

exists on a neighborhood $N' \subset N$ of $z = 0$, even if not as a single-valued function, and this is the main premise of the present study.

To make it effective still requires transformation to the framework of the natural variables, and a preliminary remark on notation may be helpful. Observe that (2) shows $\varepsilon^{-1}f$ to be a function of z, not of z/ε, and therefore by (3), of εx, rather than of x. The primes in (2), of course, denote d/dx, and thus y must be anticipated to be a function both of x and of the parametric variable

$$\varepsilon x = \xi \quad .$$

Explicitly notation to that effect is cumbersome and technically redundant, because the proofs below are for fixed $\varepsilon \neq 0$, but perception of the structure of the theory will be made easier by the convention that a notation such as $q(x)$ denotes a function that may depend also on ξ while a notation such as $\psi(\xi)$ denotes a function (such as $\varepsilon^{-1}f$) only of the modulation variable $\xi = \varepsilon x$.

Now, the ξ-image of N' will contain a disc about $\xi = 0$. Since the analysis will be local in ξ, no generality is lost by subjecting its radius E also to a bound $E(\gamma)$ specified later for the convenience of the estimates. This disc, <u>with a cut</u>, will be the ξ-domain Δ of the analysis; the corresponding x-domain D is the cut disc of radius $E/|\varepsilon|$. An intrinsic statement of our premise (3) is therefore that a branch $r(x)$ of $q^{1/2}$ must be definable as an analytic function on D so that

$$i dz/d\xi = r^{-2}$$

is integrable to $\xi = 0$.

It may be conjectured that this physical hypothesis is sufficient for a theory of irregular points of modulation. To obtain more concrete results, however, we add a second, lesser hypothesis that the function $xf(x) = (x/r)dr/dx$ which, like $\varepsilon^{-1}f$, depends only on εx, has a limit,

$$xf(\,.\ \to \gamma \in \quad \text{as} \quad \xi \to 0, \quad \text{uniformly in } \Delta \ .$$

Accordingly,

$$q^{1/2} = r(x) = x^{\gamma}\rho(\varepsilon x) \tag{4}$$

and

$$(\xi/\rho)d\rho/d\xi = xf(x) - \gamma = \phi(\xi) \tag{5}$$

is analytic on Δ and

$$\phi(\xi) \to 0 \quad \text{as} \quad \xi \to 0, \quad \text{uniformly in } \Delta \ . \tag{6}$$

In view of the restrictions to be placed on the radius E of Δ for the estimates, no further loss of generality arises from assuming $|\phi(\xi)|$ also to be bounded on Δ.

To interpret the second hypothesis, note that, apart from a constant factor,

$$\rho(\xi) = \exp \int^{\xi} \tau^{-1}\phi(\tau)d\tau \tag{7}$$

whence it follows from (6) that $\rho(\xi)$ is analytic on Δ and varies near $\xi = 0$ more slowly than any nonzero power of ξ,

$$\forall\nu > 0, \ |\xi^{\nu}\rho^{\pm 1}| \to 0 \quad \text{as} \quad \xi \to 0 \ . \tag{8}$$

The postulated limit γ is therefore the "<u>nearest power</u>" of x in $r(x) = q^{1/2}$ (and $q^{-1/2}$ is recalled to be the familiar amplitude function of the WKB-approximations for (1)). The integrability premise is now equivalent to

$$\text{Re}\ \gamma < \frac{1}{2} \tag{9}$$

with an added restriction on ρ and ϕ in case $\text{Re}\ \gamma = \frac{1}{2}$. In turn, the term "<u>mild function</u>" would seem apposite for any function $\psi(\xi)$ sharing the defining property of $\rho(\xi)$ that it is analytic on Δ and $(\xi/\psi)d\psi/d\xi \to 0$ as $\xi \to 0$, uniformly in Δ, whence also (8) then follows for ψ.

For further interpretation, note that the singular point is regular for $\varepsilon = 0$ because then $\phi \equiv 0$ and the modulation function $f(x)$ in (2) has a simple pole. The <u>irregularity function</u> $\phi(\varepsilon x)$ defined by the second hypothesis, however vaguely, identifies a diffeomorphism linking the irregular points to regular ones. A mathematical role of homotopy parameter is thereby revealed explicitly for $|\varepsilon|$.

The class of singular points of Schroedinger equations admitted by the two hypotheses includes very irregular ones, due to the unrestricted multivaluedness of $\phi(\varepsilon x)$ and hence, also of $r(x)$ and $q(z)$, in addition to all the turning points of modulation covered in the literature. It extends even the class of [7] (where only an asymptotic approximation for $|x| \to \infty$ and $|\varepsilon| \to 0$ is established) by abandonment of any restriction on how slowly $\phi(\xi) \to 0$ with ξ. For Langer's [1] class of fractional turning points, $z^{2\gamma/(2\gamma-1)}q(z)$ is analytic and nonzero at $z = 0$,

$$\phi(\xi) = \sum_{n=1}^{\infty} \tilde{\gamma}_n \, \xi^{n(1-2\gamma)} \quad ,$$

and the solutions of (1) and (2) are approximable in terms of Bessel functions.

3. The Milder Solution

The regular singular points of (2) have Frobenius exponents or indices [4, p. 150] 0 and $1 - 2\gamma$ and the auxiliary function $z(x)$ which (3) associates with each modulation equation (2) will turn out to generalize the Frobenius power $x^{1-2\gamma}$ to irregular points. Indeed, by (3) to (6) and L'Hopital's rule, as $x \to 0$ in D,

$$\begin{aligned} \frac{x}{2}\frac{dz}{dx} = \frac{-i\varepsilon x}{r^2 z} &\to -i\varepsilon \lim_{x\to 0} \frac{d(r^{-2}x)/dx}{dz/dx} \\ &= \lim_{x\to 0} (1 - 2xf) = 1 - 2\gamma \end{aligned} \qquad (10)$$

so that

$$z(x) = x^{1-2\gamma}\,\zeta(\varepsilon x) \qquad (11)$$

and $\zeta(\xi)$ is mild:

$$(\xi/\zeta)d\zeta/d\xi = \omega(\xi) \to 0 \quad \text{as} \quad \xi \to 0$$

uniformly in Δ.

The idea behind most of the estimates to follow is that such mild functions can be controlled at the expense of arbitrarily small powers. For instance, if

$$\begin{gathered} \operatorname*{lub}_{u\in(0,1)} |\phi(u\xi) + \omega(u\xi)| = \delta_1(\xi) \quad , \\ \rho(u\xi)\zeta(u\xi)u^{\delta_1}/[\rho(\xi)\zeta(\xi)] = e_1(u,\xi) \end{gathered} \tag{12}$$

then

$$|e_1| < \text{ for } u \in [0,1] \text{ and } \delta_1(\xi) \to 0 \text{ as } \xi \to 0 \tag{13}$$

because ω and $\phi \to 0$ with ξ and

$$\log e_1(u,\xi) = -\int_u^1 [\delta_1 + \phi(\tau\xi) + \omega(\tau\xi)] \frac{d\tau}{\tau}$$

has non-positive real part. A restriction $|\xi| < E_1$ with an $E_1 > 0$ can therefore assure $\delta_1(\xi) < 1$ and thereby, $\delta_1 + \operatorname{Re}\gamma < 3/2$, by (9).

Theorem 1. For $|\xi| < E(\gamma)$ so restricted that

$$M = (3/2) - \gamma - \delta_1(\xi) \tag{14}$$

has positive real part (and at worst, $|\xi| < E_1$), (2) possesses a solution

$$y_m(x) = z(x)\hat{y}(x) = x^{1-2\gamma}\zeta(\epsilon x) \sum_0^\infty \alpha_n(\epsilon x)(x/2)^{2n} \tag{15}$$

on D with z, ζ and δ_1 defined by (3), (11) and (12) and α_n given recursively by $\alpha_0 \equiv 1$ and

$$\alpha_n(\xi) = 4 \int_0^1 u^{2n-1} du \int_0^1 [e_1(\lambda,u\xi)]^2 \alpha_{n-1}(\lambda u\xi)\lambda^{2M-3+2n} d\lambda \quad .$$

Proof. By (2) and (3), $y(x)/z(x) = \hat{y}(x)$ obeys

$$\hat{y}'' + 2(f + z'/z)\hat{y}' - \hat{y} = 0 \quad , \tag{16}$$

which will be satisfied by a differentiable solution of the Volterra equation

$$\begin{gathered} \hat{y}'(x) = \int_0^x \left[\frac{r(v)z(v)}{r(x)z(x)}\right]^2 \hat{y}(v)dv \\ \hat{y}(x) = 1 + \int_0^x \hat{y}'(v)dv \quad . \end{gathered} \tag{17}$$

Let $m = \operatorname{Re} M$ and

$$\frac{da_{n+1}}{dx} = \int_0^x \left[\frac{r(v)z(v)}{r(x)z(x)}\right]^2 a_n(v)\,dv \tag{18}$$

$$a_0 \equiv 1, \quad a_n(0) = 0 \quad \text{for} \quad n \geq 1 \quad ,$$

then by (4), (11) and (12),

$$x^{-1}a'_{n+1} = \int_0^1 [e_1(u,\xi)u^{1-\gamma-\delta_1}]^2 \, a_n(ux)\,du \tag{19}$$

and the restriction on $E(\gamma)$ assures by (13) that

$$|x^{-1}a'_1| \leq (2m)^{-1}, \qquad a_1(x) = \frac{1}{4}\,x^2\alpha_1(\xi), \qquad |\alpha_1| \leq 1/m \quad ,$$

and if $a_n(x) = \alpha_n(\xi)(x/2)^{2n}$ with $|\alpha_n(\xi)| \leq k_n$, that

$$|(x/2)^{-2n-1}a'_{n+1}| \leq k_n/(m+n), \qquad a_{n+1}(x) = \alpha_{n+1}(\xi)(x/2)^{2n+2}$$

$$|\alpha_{n+1}(\xi)| \leq k_n(m+n)^{-1}(n+1)^{-1} = k_{n+1} \quad .$$

Inductively, therefore

$$a_n(x) = \alpha_n(\xi)(x/2)^{2n}, \qquad |\alpha_n(\xi)| \leq k_n = \Gamma(m)/[n!\,\Gamma(m+n)] \tag{20}$$

and these bounds show the series $\sum a'_n(x)$ of functions analytic on D to converge uniformly on compact subsets of D to a function $\hat{y}'(x)$ analytic on D. The series $\sum a_n$ converges similarly to $\hat{y}(x)$, and summation of (18) confirms that $\hat{y}(x)$ satisfies (17). The formula for $\alpha_n(\xi)$ follows from (18) and (19).

Corollaries. Since $z(0) = 0$, by (3), also

$$y_m(0) = 0 \quad . \tag{21}$$

For $\varepsilon = 0$ and $\operatorname{Re} \gamma < \frac{1}{2}$, in which case the singular point is regular, ζ and α_n are constant and (15) is precisely the Frobenius series [4, p. 149] for the solution of (2) with the property (21). (For $\operatorname{Re} \gamma = \frac{1}{2}$, the integrability premise does not admit $\varepsilon = 0$.) For $\varepsilon \neq 0$, $\zeta(\xi)$ and $\alpha_n(\xi)$ are generally multivalued on full neighborhoods of the irregular point.

L'Hopital's rule shows (Appendix I) none the less that

$$n!\alpha_n(0) = \Gamma(\tfrac{3}{2} - \gamma)/\Gamma(n + \tfrac{3}{2} - \gamma) \quad .$$

The sense in which $\hat{y}(x)$ tends to an even function of x as $\xi \to 0$ will be discussed in Section 5. Summation of the bounds (20) for (15) yields growth bounds

$$|y_m(x)/z(x)| = |\hat{y}(x)| \leq \Gamma(m)|x/2|^{1-m} I_{m-1}(|x|)$$

$$|d\hat{y}/dx| \leq \Gamma(m)|x/2|^{1-m} I_m(|x|), \quad m - 1 = \tfrac{1}{2} - \mathrm{Re}\ \gamma - \delta_1(\xi)$$

for the milder solution $y_m(x)$ in terms of modified Bessel functions [4, p. 60]. They emphasize even further how the irregular points of modulation concern generalization of Bessel functions.

4. The Stronger Solution

Theorem 2. If $\frac{1}{2} - \mathrm{Re}\ \gamma$ is not a positive integer and E is suitably restricted, then (2) has a solution

$$y_s(x) = \sum_0^\infty \beta_p(\varepsilon x)(x/2)^{2p}$$

analytic on D, with $\beta_0 \equiv 1$ and $\beta_p(\xi)$ defined recursively by (23), (29) and Lemmas 1, 2 below.

Remarks. Since $y_s(0) = 1$, $y_s(x)$ represents a stronger solution. So does $y_s(x) + ay_m(x)$ for any $a \in \mathbb{C}$, but the clearest representation of the branch structure of the singular point will be by a fundamental system (y_s, y_m) in which y_s is free of such additive traces of the branch point of y_m. Comparison with Theorem 1 shows Theorem 2 to complete such a system because $\beta_p(0)$ exists and is nonzero (Appendix II) for all $p \geq 0$. In fact, $\sum \beta_p(0)(x/2)^{2p}$ is the power series of (28) below [4, p. 60], but for $\varepsilon \neq 0$, the $\beta_p(\xi)$ are generally multivalued on a full neighborhood of the irregular point. Summation of the bounds of Lemma 2 below again yields growth bounds for y_s and y_s' in terms of modified Bessel functions.

Proof. A twice differentiable solution $y_s(x)$ of

$$r^2 y^1(x) = \int_0^x r^2 y(v)\,dv + \text{const.}$$

$$y(x) = 1 + \int_0^x y^1(v)\,dv$$

will satisfy (2). For $\mathrm{Re}\,\gamma \leq -\frac{1}{2}$, however, the normalization to $y_s(0) = 1$ is seen by (5) to require a regularization of the first integral, and different regularizations will add different multiples of the milder solution to the stronger. A procedure avoiding it is to write this Volterra equation as

$$r^2 y'(x) = \int_X^x r^2 y(v)\,dv + C_X \quad ,$$

$$C_X = \int_0^X r^2 (y - y_N)\,dv \tag{22}$$

$$y(x) = 1 + \int_0^x y'(v)\,dv$$

with a fixed regularization parameter $X \neq 0$ and

$$y_N(x) = \sum_0^N b_p(x), \qquad r^2 b'_{p+1} = \int_X^x r^2 b_p(v)\,dv$$

$$y_{-1} \equiv b_{-1} \equiv 0,\ b_0 \equiv 1, \qquad b_{p+1}(x) = \int_0^x b'_{p+1}(v)\,dv \tag{23}$$

for

$$0 \leq p \leq N \leq -\frac{1}{2} - \mathrm{Re}\,\gamma < N+1 \quad . \tag{24}$$

This implies

$$(r^2 y_N')' - r^2 y_N = -r^2 b_N \quad . \tag{25}$$

To describe the structure of y_N for $N \geq 1$, require $0 < |\varepsilon X| < E$ to assure definition of $\phi(\varepsilon x)$ for $|x| \leq |X|$ and note that

$$\operatorname*{lub}_{|\xi|<|\varepsilon X|} |\phi(\xi)| = \delta(|\varepsilon X|) \to 0 \text{ as } |\varepsilon X| \to 0 \quad , \tag{26}$$

by (6), so that $E < E_2$ with an $E_2 > 0$ will assure $\delta < 1$. Appendix II outlines a proof of

Lemma 1. For $|\xi| < E_2$ and for $1 \leq p \leq N$,

$$b_p(x) = \beta_p(\xi)(x/2)^{2p}$$

and $|\beta_p| \leq k'_p = (k')^p/p!$ with $k'(\gamma)$ independent of x and X. If $\frac{1}{2} - \operatorname{Re}\gamma$ is not a positive integer, the result remains true also for $p = N+1$ for sufficiently small $E(\gamma) > 0$.

This shows $y_N(x)$ to be a 'polynomial' in x^2 of degree N with coefficients that are bounded, but generally multivalued, functions of $\xi = \varepsilon x$. For integer $-\frac{1}{2} - \operatorname{Re}\gamma$, the bound on $|\beta_{N+1}|$ fails, indeed (Appendix II), $\beta_p(0)$ exists for $0 \leq p \leq N+1$ when $-\frac{1}{2} - \operatorname{Re}\gamma$ is not integer, but only for $0 \leq p \leq N$ when it is. For $\varepsilon = 0$, in fact, $y_N(x)$ is the sum of the first N terms of the power series of the solution

$$\lim_{\varepsilon=0} y_s(x) = \Gamma(\tfrac{1}{2} + \gamma)(x/2)^{\frac{1}{2}-\gamma} I_{\gamma-\frac{1}{2}}(x) \tag{28}$$

of (2) for $\phi(\xi) \equiv 0$.

To attack (22), consider now a sequence

$$\begin{aligned} y_n(x) &= y_N(x) + \sum_{p=N+1}^{n} b_p(x), \qquad n = N+1, N+2, \ldots \\ b'_{p+1}(x) &= \int_0^x \left[\frac{r(v)}{r(x)}\right]^2 b_p(v)\,dv, \qquad b_{p+1}(0) = 0 \end{aligned} \tag{29}$$

with b_{N+1} given by Lemma 1 and N defined by (24), so that $-\frac{1}{2} < N+1 + \operatorname{Re}\gamma$ and these integrals converge for $p \geq N+1$, by (4) to (6). Appendix II also gives a proof of

Lemma 2. If $-\frac{1}{2} - \operatorname{Re}\gamma > 0$ and not integer and if $|\xi| < E(\gamma)$ with sufficiently small $E(\gamma) > 0$, then

$$b_p(x) = \beta_p(\xi)(x/2)^{2p} \quad ,$$

$$p!|\beta_p| < p!k'_p = (k')^{N+1}\Gamma(N+1-s)/\Gamma(p-s)$$

for $p > N+2$, with

$$s = \frac{1}{2} - \operatorname{Re}\gamma + \delta_2(\xi) > 0 \quad , \tag{30}$$

$$\delta_2(\xi) = \operatorname*{lub}_{u\in(0,1)} |\phi(u\xi)| \quad .$$

For $-\frac{1}{2} < \operatorname{Re}\gamma < \frac{1}{2}$, the construction simplifies because no regularization is needed. Then $N = -1$, $y_N \equiv 0$, $b_0 \equiv 1$ and the same proof leads to Lemma 2 for $p > 1$, except that $s < 0$ and

$$p!k'_p = \Gamma(-s)/\Gamma(p-s) \quad .$$

The sequence (29) of functions analytic on D is therefore majorized by the partial sums of a power series convergent for all x and converges uniformly on compact subsets of D to a function $y_s(x)$ analytic on D, which summation of (29) shows to satisfy (22).

For the sake of completeness, information on the nature of the stronger solution for non-positive integer values of $\operatorname{Re}\gamma - \frac{1}{2}$ may be desired. For $\varepsilon = 0$, where the singular point is regular, these are the exceptional exponent differences [4, p. 150] for which the stronger solution has a logrithmic branch point. For irregular points the case $\gamma = \frac{1}{2}$ is somewhat special because the integrability premise does not then admit $\varepsilon = 0$, but assures $y_m(0) = 0$; by contrast, the construction of the stronger solution makes $y_s(0) = 1$ also for $\operatorname{Re}\gamma = \frac{1}{2}$. For $\operatorname{Re}\gamma < -\frac{1}{2}$, a different representation of the stronger solution can help to elucidate its nature.

Theorem 3. If

$$\frac{dq}{dz} = \frac{i}{\varepsilon[y_m(x)]^2}\int_0^x [r(v)]^2 b_N(v) y_m(v)\,dv \quad , \tag{31}$$

then $y_N(x) + g(x)y_m(x)$ solves (2). For non-integer $\frac{1}{2} - \text{Re}\,\gamma > 1$, the additional specification

$$\rho^2 x^{2\gamma-1}[g(x)y_m(x) - \beta_{N+1}(0)(x/2)^{2N+2}] \to 0 \quad \text{as} \quad x \to 0$$

makes $y_N + gy_m$ identical with the stronger solution of Theorem 2.

Proof. For any twice differentiable function $g(x)$, $g(x)y_m(x)$ solves

$$[r^2(gy_m)']' = r^2 gy_m + y_m^{-1}(r^2 y_m^2 g')'$$

because $y_m(x)$ satisfies (2). The definition of $dg/dx = g'(x)$ by the theorem and (3) makes $(r^2 y_m^2 g')' = r^2 y_m b_N$ and by (25), therefore, $y_N + gy_m$ solves (2) and must be a linear combination of y_m and y_s. But, $g(x)$ can be examined with the help of Theorem 1 and Lemma 1 (Appendix II):

Lemma 3. For non-integer $\frac{1}{2} - \text{Re}\,\gamma > 1$,

$$x^{-2N-2} g(x)y_m(x) \to 4^{-N-1}\beta_{N+1}(0) \quad \text{as} \quad x \to 0 \quad ,$$

with N defined by (24) and $\beta_{N+1}(0)$, by (A7).

Since $N \geq 0$ and $y_N(0) = b_0 = 1$, it follows from (21) that $y_N + gy_m$ represents a stronger solution y of (2) normalized to $y(0) = 1$. From (24), moreover, $2N+2 < 1 - 2\text{Re}\,\gamma < 2N+4$, while Theorem 1 shows $y_m(x) = O(x^{1-2\gamma}\zeta)$, so by Lemma 3, $y_N + gy_m$ represents the same solution as Theorem 2 up to terms $o(x^{2N+2})$. The choice of additive constant in $g(x)$ specified in Theorem 3 extends this agreement sufficiently, by (10) and Theorem 1, to preclude a difference of a nonzero multiple of $y_m(x)$ between the stronger solutions of the two theorems.

While Lemma 3 holds only on strips of it, (31) is defined by Theorem 1 and Lemma 1 in the whole left half-plane of $\text{Re}\,\gamma - \frac{1}{2}$ and there depends analytically on γ, by (2) and (5), when x, ε and $\phi(\xi)$ are fixed so that $y_m(x) \neq 0$. In the strips, the proof of Lemma 3 (Appendix II) shows

$$(x/g)\,dg/dx \to 2N + 2\gamma + 1 = 2\nu \quad \text{as} \quad x \to 0 \quad ,$$

and as $\text{Re}\,\nu \uparrow 0$ so that the righthand strip boundary is approached, $g(x)$

is seen to approach a function varying less than any nonzero power of x (although it approaches a strictly mild function only as $\nu \to 0$, $\gamma - \frac{1}{2} \to$ integer). Thus $g(x)$ extends the logarithms of Frobenius [4, p.150] to more general, irregular points; in the special case where $\rho(\xi) \to \rho(0) \neq 0$, as for a fractional turning point and in particular, for a regular point, equation (A11) below shows $g(x)$ to have a purely logarithmic branch point when $\gamma - \frac{1}{2}$ is a negative integer.

5. Symmetry Bounds

The coefficient of (10) depends continuously on ε, at any fixed $x \neq 0$, by (3) to (6), and as $\varepsilon \to 0$, the equation approaches a form of Bessel's with solution

$$\lim_{\varepsilon=0} y(x) = \Gamma(\tfrac{3}{2} - \gamma)(x/2)^{\gamma-\frac{1}{2}} I_{\frac{1}{2}-\gamma}(x)$$

normalized to $y(0) = 1$. At the same time, the restriction on $|x|$ disappears that had been needed to respect the merely local definition of $\phi(\xi)$, so that D expands to the whole, cut plane of x. As $\xi \to 0$, therefore, $y(x) = y_m/z$ tends to an even analytic function of x on any compact $\subset \mathbb{C}$. The same is true of $y_s(x)$, by (28), as long as $\frac{1}{2} - \mathrm{Re}\,\gamma$ is not an integer. It is of considerable interest that these remarks can be sharpened to detailed bounds on the oddness of the solutions for nonzero $\xi \in \Delta$, which can serve as foundation of a theory of asymptotic connection [3]:

Theorem 4. For x and $xe^{-\pi i}$ in D and $E(\gamma)$ restricted as for Theorem 1,

$$|y(x) - y(xe^{-\pi i})| \leq \delta_m(|\xi|)\Gamma(m)|x/2|^{2-m} I_m(|x|)$$

and $\delta_m \to 0$ as $|\xi| \to 0$.

Here $m = (3/2) - \mathrm{Re}\,\gamma - \delta_1(\xi)$, as in Theorem 1, with δ_1 defined by (12), and I_m denotes again the modified Bessel function [4, p. 60]; a proof is given in Appendix I. In case the relation of the order m of the Bessel function to the variable ξ be considered more confusing than useful, it is of course permissible to proceed from (12) with δ_1 replaced throughout the proofs by

$$\delta_*(\gamma) = \operatorname*{lub}_{\xi \in \Delta} \delta_1(\xi)$$

to obtain Theorem 1 with $M = (3/2) - \mathrm{Re}\,\gamma - \delta_*$ and Theorem 4, with $m = 3/2 - \mathrm{Re}\,\gamma - \delta_*$.

For fixed $\xi \in \Delta$, the oddness of $y_m/z = \hat{y}(x)$ can therefore grow at most exponentially with $|x|$.

The worst part of our story is a proof (Appendix II) of a similar bound on the first N terms of the stronger solution for $\mathrm{Re}\,\gamma < -3/2$ (for $\mathrm{Re}\,\gamma > -3/2$, $y_N(x)$ is strictly even because $y_{-1} \equiv 0$, $y_0 \equiv 1$):

Lemma 4. For x and $xe^{-\pi i}$ in D and sufficiently restricted $E(\gamma)$,

$$|y_N(x) - y_N(xe^{-\pi i})| < \delta_s'(|\xi|) \sum_1^N [k'(\gamma)|x^2/4|]^p/(p-1)!$$

and $\delta_s' \to 0$ as $|\xi| \to 0$.

Finally, the bound is extended in Appendix II to the stronger solution:

Theorem 5. For non-integer $\frac{1}{2} - \mathrm{Re}\,\gamma$, x and $xe^{-\pi i}$ in D and sufficiently restricted $E(\gamma)$,

$$|y_s(x) - y_s(xe^{-\pi i})| < C(\gamma)\delta_s(|\xi|)|x/2|^{2+s} I_{-s}(|x|)$$

and $\delta_s \to 0$ as $|\xi| \to 0$.

Here s is again defined by (30), but the same proof also yields the Theorem a fortiori with

$$s = -\frac{1}{2} - \mathrm{Re}\,\gamma + \operatorname*{lub}_{\xi \in \Delta} |\phi(\xi)| \quad .$$

For integer $\frac{1}{2} - \mathrm{Re}\,\gamma > 0$, however, Theorem 3 and Lemma 3 show Theorem 5 to fail.

Appendix I

To compute the limit of $\alpha_n(\xi)$ as $\xi \to 0$ in Δ, note that $\alpha_0 \equiv 1$ by definition and suppose that $\alpha_{n-1}(0)$ exists for some $n \geq 1$. By L'Hopital's rule applied to (20),

$$\alpha_n(\xi) \to (2^{2n-1}/n) \lim_{x\to 0} [a_n'(x)/x^{2n-1}] \quad \text{as} \quad \xi \to 0$$

if that limit exists. In turn, by the same rule applied to (18),

$$x^{1-2n} a_n'(x) \to 2^{2-2n} \lim[\int_0^x r^2 z^2 v^{2n-2} \alpha_{n-1} dv/(x^{2n-1} r^2 z^2)]$$

$$= 2^{2-2n} \lim[\alpha_{n-1}/(2n-1 + 2xr'/r + 2xz'/z)]$$

and by (4) to (6) and (10),

$$\alpha_n(0)/\alpha_{n-1}(0) = n^{-1}(n + \tfrac{1}{2} - \gamma)^{-1} \quad .$$

Proof of Theorem 4. From (18) if $e^{-\pi i}$ be abbreviated by j and if x and $jx \in D$,

$$a_{n+1}'(x) + a_{n+1}'(jx) = \int_0^x \{[\frac{r(v)z(v)}{r(x)z(x)}]^2 a_n(v) - [\frac{r(jv)z(jv)}{r(jx)z(jx)}]^2 a_n(jv)\}dv$$

and by (4), (11) and (12),

$$r(v)z(v)/[r(x)z(x)] = e_1(u,\xi)u^{1-\gamma-\delta_1} \quad .$$

By (5) and (11),

$$\frac{r(jv)z(jv)}{r(v)z(v)} \cdot \frac{r(x)z(x)}{r(jx)z(jx)} = e^{\lambda(u,\xi)} \quad ,$$

$$\lambda(u,\xi) = \int_{u\xi}^{ju\xi} \frac{\mu(\sigma)}{\sigma} d\sigma + \int_{j\xi}^{\xi} \frac{\mu(\sigma)}{\sigma} d\sigma \quad ,$$

$$\mu(\xi) = \xi d(\log \rho\zeta)/d\xi = \phi(\xi) + \omega(\xi) \quad .$$

Therefore

$$\begin{aligned} a_{n+1}'(x) + a_{n+1}'(jx) = x \int_0^1 e_1^2 u^{2-2\gamma-2\delta_1} [a_n(ux) - a_n(jux) \\ + a_n(jux)\{1 - \exp 2\lambda(u,\xi)\}]du \end{aligned} \tag{A1}$$

and since ρ and ζ are mild, both $\mu(\xi)$ and, for $u \in [0,1]$,

$$\lambda(u,\xi) = i \int_{-\pi}^{0} \{\mu(|\xi|e^{i\theta}) - \mu(|u\xi|e^{i\theta})\}d\theta$$

tend to zero with ξ, so that also

$$\delta_m(|\xi|) = \underset{|t|<|\xi|}{\text{lub}} \; \underset{u\in[0,1]}{\text{lub}} \; |1 - \exp 2\lambda(u,t)| \to 0 \quad \text{as} \quad |\xi| \to 0 \quad .$$

Since $a_0 \equiv 1$, (A1) gives $|x^{-1}[a_1'(x) + a_1'(jx)]| < \frac{1}{2}\,\delta_m/m$ with $m = \text{Re } M = (3/2) - \text{Re } \gamma - \delta_1$ as in Theorem 1, and it now follows recursively from (A1) and (20) that

$$|(x/2)^{1-2n}[a_n'(x) + a_n'(jx)]| < n^2\delta_m k_n$$

$$|(x/2)^{-2n}\,[a_n(x) - a_n'(jx)]| < n\;\delta_m k_n$$

and Theorem 4 follows from (20) and Theorem 1.

Appendix II

Proof of Lemma 1. From (23), (4) and (5),

$$b_1'(x) = r^{-2} \int_X^x r^2 dv = x \int_{X/x}^1 [e_2(u,\xi)]^2 u^{2\gamma+2\delta} du \quad ,$$
$$e_2(u,\xi) = \exp \int_\xi^{u\xi} [\phi(\tau) - \delta] \frac{d\tau}{\tau} \quad . \tag{A2}$$

Choose $\arg X$ so that $|\arg \xi - \arg(\varepsilon X)| \leq \pi$ for all $\xi \in \Delta$ (Figure 1) and the path in (A2), so that $u\xi$ moves on an arc of constant modulus from εX to $\xi' = |\varepsilon X|\exp(i \arg \xi)$ (Figure 1) and thence, radially inward to ξ. For $u\xi$ on this path P,

$$|e_2| \leq \exp(\pi\delta)$$

with δ defined by (26). For fixed ξ, u moves also along a path partly at constant modulus and partly radial, and since $N \geq 1$ confines our attention to $\operatorname{Re} \gamma \leq -3/2$, the restriction $|\xi| < E_2$ assures

$$s_1 = \operatorname{Re} S_1 > 0 \quad \text{for} \quad S_1 = -\gamma - \frac{1}{2} - \delta \tag{A3}$$

and from (A2)

$$x^{-1} b_1'(x) = \int_1^{x/X} [e_2(\tfrac{1}{t},\xi)]^2 t^{2S_1-1} dt$$

with path again partly radial and partly around (at most half of) the unit circle, and it follows for $|x/X| \leq 1$ that

$$|x^{-1} b_1'(x)| \leq e^{2\pi\delta} \int_0^1 |t^{2s_1-1} dt| \leq \frac{1}{2} k_0' \quad .$$

By (23), therefore, $|4x^{-2} b_1(x)| \leq k_0'$. If $b_{p-1}(x) = \beta_{p-1}(\xi)(x/2)^{2p-2}$, moreover, then again by (23),

$$\left(\frac{2}{x}\right)^{2p-1} b_p'(x) = 2 \int_{X/x}^1 e_2^2 \beta_{p-1}(u\xi) u^{2(\gamma+\delta+p-1)} du$$
$$= 2 \int_1^{x/X} e_2^2 \beta_{p-1}(\xi/t) t^{2S_1+1-2p} dt$$

and the restriction $|\xi| < E_2$ assures for $p \leq N$ that $2s_1 + 1 - 2p \geq 1 - 2\delta > -1$, by (24) and (A3). Therefore

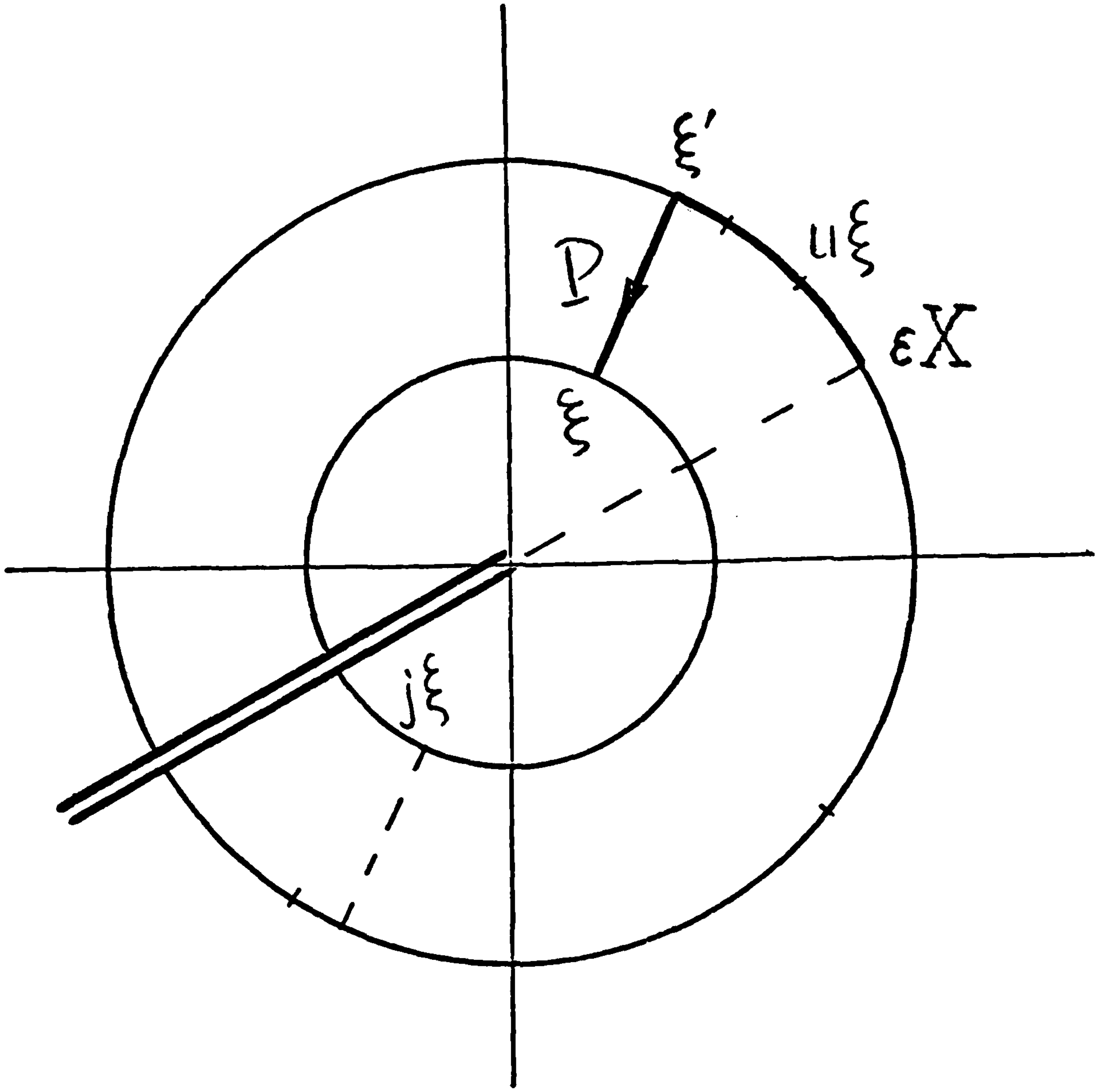

Figure 1

$$|(2/x)^{2p-1} b_p'(x)| \le 2e^{2\pi\delta} \operatorname*{lub}_{|\xi| \le |\varepsilon x|} |\beta_{p-1}| \int_0^1 |t^{2s_1+1-2p} dt|$$

and if $|\beta_{p-1}| \le k'_{p-1}$ and

$$k' = 2e^{2\pi\delta} \max_{1 \le n \le N-1} \int_0^1 |t^{2s_1-1-2n} dt|$$

with path as before, then

$$(2/x)^{2p} b_p(x) = \beta_p(\xi), \qquad |\beta_p| \le k' k'_{p-1}/p = k'_p \quad , \tag{A4}$$

and for $p \le N$, the Lemma follows by induction.

For non-integer $\frac{1}{2} - \mathrm{Re}\,\gamma$, (24) implies $N + \mathrm{Re}\,\gamma < -\frac{1}{2}$, so by (A3), $2s_1 - 1 - 2N + 2\delta = -2(N+1 + \mathrm{Re}\,\gamma) > -1$ and for fixed $\mathrm{Re}\,\gamma$, (26) shows an $E(\gamma) > 0$ to exist which makes also $2s_1 - 1 - 2N > -1$. The integral for $(2/x)^{2p-1} b_p'(x)$ then exists also for $p = N+1$ and (A4) remains valid with k' maximized over $1 \le n \le N$.

While the rough bound (A4) suffices for present purposes, it underestimates the decrease of $|\beta_p|$ with increasing p. A more representative indication of it is obtained by computing $\beta_p(0)$: As $\xi \to 0$, by Lemma 1 and L'Hopital's rule,

$$\beta_p(\xi) \to 2^{2p-1} p^{-1} \lim_{x \to 0} [x^{1-2p} b_p'(x)]$$

if that limit exists, and from (23), (5) and (6),

$$x^{1-2p} b_p'(x) = (r^{-2} x^{1-2p}) \int_X^x r^2 b_{p-1} dv$$

$$+ 2^{2-2p} \lim_{x \to 0} [\beta_{p-1}(\xi)/(2p - 1 + 2xr'/r)]$$

$$= 2^{1-2p} \beta_{p-1}(0)/(p + \gamma - \tfrac{1}{2})$$

if that exists. Since $\beta_0 \equiv 1$, it follows that

$$\beta_p(0) = p^{-1}(p + \gamma - \tfrac{1}{2})^{-1} \beta_{p-1}(0) \quad ,$$

provided $\gamma - \frac{1}{2}$ is not a negative integer, and then by induction

$$p!\ \beta_p(0) = \Gamma(\tfrac{1}{2} + \gamma)/\Gamma(p + \tfrac{1}{2} + \gamma) \tag{A6}$$

for all $p \geq 0$. In any case, (24) implies $N + \frac{1}{2} + \text{Re}\,\gamma < 0$ so that the induction yields

$$p!\,\beta_p(0) = (-1)^p\Gamma(\tfrac{1}{2} - \gamma - p)/\Gamma(\tfrac{1}{2} - \gamma) \quad \text{for} \quad 0 \leq p \leq N \tag{A7}$$

even when $\gamma - \frac{1}{2}$ is a negative integer, but $\beta_{N+1}(0)$ is then seen not to exist.

Proof of Lemma 2. By (6),

$$\delta_2(\xi) \leq \delta_\Delta = \operatorname*{lub}_{\xi \in \Delta} |\phi(\xi)| \to 0 \quad \text{as} \quad E \to 0 \tag{A8}$$

and for fixed $\text{Re}\,\gamma < -\frac{1}{2}$, a sufficient restriction of $E(\gamma)$ assures $s > 0$. If again

$$b_{p-1}(x) = \beta_{p-1}(\xi)(x/2)^{2p-2}, \qquad |\beta_{p-1}| \leq k'_{p-1}$$

for some $p \geq N+2$, then from (29), (4) and (7),

$$b'_p(x) = 2(x/2)^{2p-1} \int_0^1 [e_3(u,\xi)]^2 \beta_{p-1}(u\xi)u^{2(\gamma-\delta_2+p-1)}\,du \quad ,$$

$$e_3(u,\xi) = u^{\delta_2}\rho(u\xi)/\rho(\xi) = \exp \int_1^u [\delta_2(\xi) + \phi(\tau\xi)]\,\frac{d\tau}{\tau}$$

and by (30),

$$|e_3(u,\xi)| \leq 1 \quad \text{for} \quad u \in [0,1] \quad . \tag{A9}$$

For $p \geq N+2$, (24) implies $p-1 > -\frac{1}{2} - \text{Re}\,\gamma$ and therefore also $\text{Re}\,\gamma - \delta_2 + p - \frac{1}{2} = p - 1 - s > 0$ for fixed γ and sufficiently small $|\xi| > 0$. Hence,

$$|(x/2)^{1-2p} b'_p| \leq k'_{p-1}/(p-1-s)$$

and by (29),

$$|(2/x)^{-2p} b_p| \leq k'_{p-1}/[p(p-1-s)] = k'_p \tag{A10}$$

and Lemma 2 follows inductively from the bound for b_{N+1} of Lemma 1.

Proof of Lemma 3. By Lemma 1 and (3), (4) and (15), the definition (31) may be written, with $N + \frac{1}{2} + \gamma = \nu$, as

$$\frac{x^{1-2\nu}}{\rho^2}\frac{dg}{dx} = 2^{-2N}\left(\frac{x}{r^2 z\hat{y}}\right)^2 x^{-2N-2}\int_0^x \frac{r^2 z}{v}\beta_N \hat{y}\, v^{2N+1}dv$$

$$+ 2^{-2N-1}\frac{i}{\varepsilon}\frac{1-2\gamma}{N+1}\beta_N(0) = C_N \neq 0 \quad \text{as} \quad x \to 0 \quad , \tag{A11}$$

by (10), (17) and (A7). Since (24) implies $-1 < \mathrm{Re}\ \nu \leq 0$ and since ρ is mild, dg/dx is seen not to be integrable to $x = 0$ unless $\mathrm{Re}\ \nu = 0$ and $\rho(\xi) \to 0$ as $\xi \to 0$.

For $\mathrm{Re}\ \nu < 0$, L'Hopital's rule may therefore be applied to $\rho^2 x^{2\nu}/g$ and yields, by (A11), (5) and (6),

$$\rho^2 x^{2\nu}/g \to \lim \frac{2x\rho^{-1}dg/dx + 2\nu}{\rho^{-2}x^{1-2\nu}dg/dx} = 2\nu/C_N$$

as $x \to 0$, and by (A11), $(x/g)dg/dx \to 2\nu$, which therefore represents the nearest power in $g(x)$. It then follows from (4), (10), (11), (A6) and (A11) that

$$x^{-2N-2}g(x)y_m(x) = \zeta\hat{y}x^{-2\nu}g$$

$$\to \frac{\varepsilon C_N}{2i\nu(1-2\gamma)} = 4^{-N-1}\beta_{N+1}(0)$$

as $x \to 0$.

Proof of Lemma 4. From (23), for $1 \leq p \leq N$ defined by (24), and again with $\exp(-\pi i) = j$,

$$b_p'(v) + b_p'(jv) = \int_X^v \left[\frac{r(t)}{r(v)}\right]^2 b_{p-1}(t)dt + \left(\int_X^{jX} + \int_{jX}^{jv}\right)\left[\frac{r(s)}{r(jv)}\right]^2 b_{p-1}(s)ds$$

$$= (v/2)^{2p-1}(I_1 + I_2 + I_3) \quad ,$$

$$I_1(v) = \left(\frac{2}{v}\right)^{2p-1}\int_X^v \left[\frac{r(t)}{r(v)}\right]^2\{b_{p-1}(t) - b_{p-1}(jt)\}dt \quad ,$$

$$I_2(v) = \left(\frac{2}{v}\right)^{2p-1}\int_X^v \left[\frac{r(t)}{r(v)}\right]^2\left\{1 - \left[\frac{r(jt)}{r(jv)}\frac{r(v)}{r(t)}\right]^2\right\}b_{p-1}(jt)dt \quad ,$$

$$I_3(v) = \left(\frac{2}{v}\right)^{2p-1}\int_X^{jX} \left[\frac{r(t)}{r(jv)}\right]^2 b_{p-1}(t)dt \quad .$$

To obtain bounds on these integrals, note from (4) and (5) that, if εv, εuv, $j\varepsilon v$ and $j\varepsilon uv$ are all in Δ,

$$\log \frac{r(juv)r(v)}{r(uv)r(jv)} = \int_{\varepsilon uv}^{\varepsilon v} \frac{\phi(\tau)-\phi(j\tau)}{\tau}\, d\tau$$

$$= \int_j^1 \frac{\phi(\varepsilon v\sigma)-\phi(\varepsilon uv\sigma)}{\sigma}\, d\sigma = \theta(u,\varepsilon v)$$

say, and $|\phi| < \delta(|\varepsilon X|)$, by (26). If εv and εuv lie on the path P of Figure 1 from ξ to εX, then the integral may be evaluated along the appropriate part of P and from the first form of θ, the circular part is seen to contribute at most $2\pi\delta$ to $|\theta|$. From the second form, the contribution of the radial part is seen to have the same bound, and together

$$|\theta| < 4\pi\delta \quad ,$$

$$\delta_3(|\varepsilon X|) = \underset{|\xi|<|\varepsilon X|}{\text{lub}} \ \underset{P}{\text{lub}} \ |1 - e^{2\theta}| \to 0 \quad \text{as} \quad |\varepsilon X| < 0 \quad . \tag{A12}$$

Moreover, if $\rho(\varepsilon s)/\rho(j\varepsilon v) = e_4(\varepsilon s,\varepsilon v)$ is considered for $\varepsilon v \in \Delta$ and $s = |X|\exp i\sigma,\ 0 > \sigma > -\pi$, then (Figure 1)

$$\log e_4 = \int_{j\varepsilon v}^{|\varepsilon v|\exp i\sigma} \phi(\lambda)\, \frac{d\lambda}{\lambda} + \int_{|\varepsilon v|}^{|\varepsilon X|} \phi(\tau e^{i\sigma})\, \frac{d\tau}{\tau}$$

and by (30), (26) and (6),

$$|e_4| < e^{\pi\delta_4}\, |X/v|^{\delta} \quad ,$$

$$\delta_4(|\xi|) = \underset{\arg \xi}{\text{lub}}\ \delta_2(\xi) \to 0 \quad \text{as} \quad |\xi| \to 0 \quad . \tag{A13}$$

From Lemma 1, $b_{p-1}(x) - b_{p-1}(jx) = \tilde{\beta}_{p-1}(\xi)(x/2)^{2p-2}$, and if it be now supposed that $|\tilde{\beta}_{p-1}(\xi)| < (p-1)\lambda_{p-1}k'_{p-1}$, then by (4) and (A2),

$$|I_1| = 2|\int_{X/v}^1 [e_2(u,\varepsilon v)]^2\, \tilde{\beta}_{p-1}(\varepsilon uv)u^{2(\gamma+\delta+p-1)}du|$$

$$< 2e^{2\pi\delta}(p-1)\lambda_{p-1}k'_{p-1} \int_{X/v}^1 |u^{2p-2S_1-3}du|$$

with S_1 defined by (A3), so that again $\mathrm{Re}\ S_1 + 1 - p > 1 - \delta > 0$, by (24) and if $E(\gamma)$ is restricted as for the first part of Lemma 1. As in the proof of that Lemma, therefore, it is deduced that

$$|I_1| < (p-1)\lambda_{p-1}k'\ k'_{p-1} \quad .$$

Similarly, by (A12) and as in the proof of Lemma 1, if $k'_p = \text{lub}\,|\beta_p|$ as there defined,

$$|I_2| = 2\left|\int_{X/x}^{1} e_2^2\{1 - e^{2\theta(u,\varepsilon v)}\}\beta_{p-1}(j\varepsilon uv)u^{2(\gamma+\delta+p-1)}du\right|$$

$$\leq k'\delta_3 k'_{p-1} \quad .$$

With path at constant $|t| = |X|$,

$$I_3 = 2\int_{X/v}^{jX/v} j^{-2\gamma}[e_4(\varepsilon t,\varepsilon v)]^2\beta_{p-1}(\varepsilon t)(t/v)^{2(\gamma+p-1)}d(t/v)$$

and again with $s_1 = \text{Re}\,S_1 = -\text{Re}\,\gamma - \frac{1}{2} - \delta$,

$$|I_3| \leq 2\pi e^{2\pi\delta_4} k'_{p-1}\,|v/X|^{2(s_1+1-p)} \quad .$$

Therefore

$$b_p(x) - b_p(jx) = \int_0^x \sum_1^3 I_i(v)(\tfrac{v}{2})^{2p-1}dv = \tilde{\beta}_p(\xi)(\tfrac{x}{2})^{2p}$$

with

$$|\tilde{\beta}_p(\xi)| = 2\left|\int_0^1 \sum_1^3 I_i(ux)u^{2p-1}du\right|$$

$$\leq [(p-1)\lambda_{p-1} + \delta_3]\,\frac{k'}{p}\,k'_{p-1} + \frac{2\pi}{s_1+1}\,e^{2\pi\delta_4}k'_{p-1}|x/X|^{2(s_1+1-p)}$$

$$= k'_p[(p-1)\lambda_{p-1} + \delta_3 + \frac{2\pi}{s_1+1}\,e^{2\pi\delta_4}\,\frac{p}{k'}\,|x/X|^{2(s_1+1-p)}]$$

by (A4). Since $\lambda_0 = 0$ because $b_0(x) \equiv 1$, it follows recursively for $1 \leq p \leq N$ that

$$b_p(x) - b_p(jx) = \tilde{\beta}_p(\xi)(x/2)^{2p}, \qquad |\tilde{\beta}_p| \leq p\lambda_p k'_p \tag{A14}$$

$$\lambda_p \leq \delta_3 + \frac{2\pi}{k'(s_1+1)}\,e^{2\pi\delta_4}\left|\frac{x}{X}\right|^{2(s_1+1-p)}[1 + \frac{p-1}{p}\,|\tfrac{x}{X}|^2$$

$$+ \frac{p-2}{p}\,|\tfrac{x}{X}|^4 + \cdots + \frac{1}{p}\,|\tfrac{x}{X}|^{2p-2}]$$

where it should be recalled from (24) and Lemma 1 that

$$0 < 1 - \delta \leq s_1 + 1 - N < 2 - \delta \quad \text{and} \quad k'_p = [k'(\gamma)]^p/p!$$

Finally,

$$|y_N(x) - y_N(jx)| = \left|\sum_1^N \tilde{\beta}_p(\xi)(\tfrac{x}{2})^{2p}\right| \leq \sum_1^N p\lambda_p k'_p|\tfrac{x}{2}|^{2p} \quad . \tag{A15}$$

To prove Lemma 4, it therefore remains to establish a bound δ_s' on λ_p which tends to zero with $|\xi|$. For sufficiently small $|x/X|$,

$$\lambda_p < \delta_3(|\epsilon X|) + \frac{4\pi}{k'(s_1+1)} |x/X|^{2(s_1+1-p)} \qquad (A16)$$

by (A13), and $E(\Upsilon)$ has been restricted so that $s_1 + 1 - p \geq 1 - \delta > 0$ for $1 \leq p \leq N$, with δ defined by (26). The requirements on the regularization parameter $|X|$ used so far are that it be independent of x and that $|\xi| < |\epsilon X| < E(\Upsilon)$, the radius of the cut disc Δ on which the estimates hold. For fixed Υ and $E(\Upsilon)$, envisage now a sequence of subdiscs about $\xi = 0$ of radius $|\epsilon X_n| < E(\Upsilon)$ and tending to zero as $n \to \infty$, and consider any point sequence $\{\xi_i\} \subset \Delta$ such that $|\xi_i| \to 0$ as $i \to \infty$. Given n, the n-th subdisc will contain all ξ_i with sufficiently large i and there is a subsequence $\{n(i)\}$ wuch that $n(i) \to \infty$ and $|x_i/X_{n(i)}| \to 0$ as $i \to \infty$. For all $|\xi| \leq |\xi_i|$, regularization can be achieved with the choice $|X_{n(i)}|$ for $|X|$, and the bounds (A12) and (A16) then imply a sequence $\{\delta_i\}$ decreasing towards zero as $i \to \infty$ and such that $\lambda_p < \delta_i$ for $|\xi| \leq |\xi_i|$ and $1 \leq p \leq N$. A monotone positive function $\delta_s'(|\xi|)$ therefore exists such that

$$\lambda_p < \delta_s'(|\xi|) \text{ for } 1 \leq p \leq N \text{ and } \delta_s'(|\xi|) \to 0 \text{ as } |\xi|/E \to 0 \ . \qquad (A17)$$

Proof of Theorem 5. By Lemma 1, the computation of $b_p(x)$ for $p \leq N$ from (23) extends to $p = N+1$ when $\frac{1}{2} - \mathrm{Re}\,\Upsilon$ is not integer. In that case, (24) implies $-\frac{1}{2} - \mathrm{Re}\,\Upsilon = N+\Lambda$ with $\Lambda(\Upsilon) > 0$, and for sufficiently small $|\epsilon X|$, $s_1 - N = \Lambda - \delta > 0$, by (26), and the estimate (A14) then extends also to $p = N+1$. For $p \geq N+2$, on the other hand, b_p is found from (29) and Lemma 2, so that then

$$b_p'(v) + b_p'(jv) = \int_0^v \{[\frac{r(t)}{r(v)}]^2 b_{p-1}(t) - [\frac{r(jt)}{r(jv)}]^2 b_{p-1}(jt)\}dt \ ,$$

again with $j = \exp(-\pi i)$. As in the proof of Lemma 2,

$$r(t)/r(v) = (t/v)^{\gamma-\delta_2} e_3(t/v,\varepsilon v) \quad ,$$

and as in the proof of Lemma 4,

$$\log \frac{r(jt)r(v)}{r(t)r(jv)} = \int_j^1 \frac{\phi(\varepsilon v\sigma)-\phi(\varepsilon t\sigma)}{\sigma}\, d\sigma = \theta(t/v,\varepsilon v) \quad ,$$

and if v and jv are in D and $t/v = u \in [0,1]$, then this integral may be evaluated at constant $|\sigma|$ to see by (30) and (6),

$$|\theta(u,\varepsilon v)| \leq 2\pi\delta_2(\varepsilon v) \quad ,$$

$$\operatorname*{lub}_{|v|\leq|x|}\ \operatorname*{lub}_{u\in[0,1]} |1 - \exp 2\theta(u,\varepsilon v)| = \delta_s''(|\xi|) \to 0 \quad \text{as} \quad |\xi| \to 0 \quad . \tag{A18}$$

Now suppose

$$b_{p-1}(x) - b_{p-1}(jx) = \tilde{\beta}_{p-1}(\xi)(x/2)^{2p-2}, \qquad |\tilde{\beta}_{p-1}| \leq (p-1)\lambda_{p-1}k'_{p-1} \quad ,$$

which is known for $p-1 = N+1$ from the extension of (A14), then since

$$b_p'(v) + b_p'(jv) = v\int_0^1 e_3^2\{b_{p-1}(uv) - e^{2\theta(u,\varepsilon v)}b_{p-1}(juv)\}u^{2\gamma-2\delta_2}du$$

and by Lemma 2, for $p \geq N+2$,

$$\left(\frac{v}{2}\right)^{1-2p}[b_p'(v) + b_p'(jv)] = 2\int_0^1 e_3^2\{\tilde{\beta}_{p-1} + (1 - e^{2\theta})\beta_{p-1}\}u^{2(p-1+\gamma-\delta_2)}du \quad ,$$

where again $p - \frac{1}{2} - \delta_2 + \operatorname{Re}\gamma = p - 1 - s > 0$ for fixed γ and sufficiently small $|\xi| > 0$, by (24) and (30). By (A9), (A10) and Lemma 2, therefore

$$|(v/2)^{1-2p}[b_p'(v) + b_p'(jv)]| \leq pk_p'[(p-1)\lambda_{p-1} + \delta_s''] \quad ,$$

$$b_p(x) - b_p(jx) = \tilde{\beta}_p(\xi)(x/2)^{2p} \quad ,$$

$$|\tilde{\beta}_p| \leq k_p'[(p-1)\lambda_{p-1} + \delta_s''] \quad .$$

It now follows recursively from (A14), (A17) and (A18) that

$|\tilde{\beta}_p| \leq \delta_s(|\xi|)pk_p'$ for all $p \geq 1$, with

$$\delta_s(|\xi|) = \max[\delta_s'(|\xi|),\delta_s''(|\xi|)] \to 0 \quad \text{as} \quad |\xi| \to 0 \quad .$$

From Theorem 2, therefore

$$|y_s(x) - y_s(jx)| \leq \delta_s(|\xi|)\sum_1^\infty pk_p'|x/2|^{2p}$$

with k'_p given by Lemmas 1 and 2 for $p \leq N+1$ and $p \geq N+2$, respectively, so that

$$\sum_1^\infty pk'_p|x/2|^{2p} \leq C(\gamma)|x/2|^2 \sum_0^\infty |x/2|^{2n}/[n!\Gamma(n+1-s)]$$

for some $C(\gamma)$ independent of x.

For $\frac{1}{2} > \mathrm{Re}\ \gamma > -\frac{1}{2}$, the proof simplifies because Lemma 4 is not needed, and the same result is obtained with $C(\gamma) = \Gamma(-s)$ and $\delta_s = \delta''_s$.

REFERENCES

1. R. E. Langer, On the asymptotic solutions of ordinary differential equations, with an application to the Bessel functions of large order, Trans. Amer. Math. Soc. 33 (1931), 23-64.
2. G. Frobenius, Ueber die integration der linearen differentialgleichungen durch reihen, J. Reine Angew. Math. 76 (1873), 214-235.
3. R. E. Meyer and J. F. Painter, Connection for wave modulation, to appear.
4. F. W. J. Olver, "Asymptotics and Special Functions", Academic Press, New York, 1974.
5. F. W. J. Olver, Second-order linear differential equations with two turning points, Phil. Trans. Roy. Soc. London A278 (1975), 137-174.
6. R. E. Meyer and E. J. Guay, Adiabatic variation, Part III, A deep mirror model, Z. Angew. Math. Phys. 25 (1974), 643-650.
7. J. F. Painter and R. E. Meyer, Turning-point connection at close quarters, SIAM J. Math. Anal. (1981).

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The local theory of singular points is extended to a large class of linear, second-order, ordinary differential equations which can be physical Schroedinger equations or govern the modulation of real oscillators or waves. In addition to Langer's fractional turning points, such equations admit highly irregular points at which the coefficients of the differential equation can be almost arbitrarily multivalued. Genuine coalescence of singular points, however, is not considered. A local representation of solution structure is established which generalizes Frobenius' method of power series. Some results on solution symmetry have striking, global implications in the shortwave limit.

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